

Riesz Bases for p -Subordinate Perturbations of Normal Operators

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Abstract. For p -subordinate perturbations of unbounded normal operators, the change of the spectrum is studied and spectral criteria for the existence of a Riesz basis with parentheses of root vectors are established. A Riesz basis without parentheses is obtained under an additional a priori assumption on the spectrum of the perturbed operator. The results are applied to two classes of block operator matrices.

Keywords. Perturbation theory, subordinate perturbation, Riesz basis, eigenvector expansion, spectrum.

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1 Introduction

Since for non-normal operators there is no analogue of the spectral theorem, the existence of a Riesz basis (possibly with parentheses) of root vectors is an important property: it allows e.g. the construction of non-trivial invariant subspaces and yields spectral criteria related to semigroup generation. For a class of non-normal perturbations of normal operators we establish different conditions in terms of the spectrum which imply the existence of such Riesz bases. Our assumptions on the multiplicities of the eigenvalues are weaker than in classical perturbation theorems.

We consider an unbounded operator $T = G + S$ on a Hilbert space where G is normal with compact resolvent and S is p -subordinate to G , i.e.

$$\|Su\| \leq b\|u\|^{1-p}\|Gu\|^p \quad \text{for all } u \in \mathcal{D}(G)$$

where $p \in [0, 1]$ and $b \geq 0$. In Theorem 6.1 we prove that T admits a Riesz basis with parentheses of root vectors if the eigenvalue multiplicities of G satisfy a certain asymptotic growth condition. This growth condition is weaker than the one in a similar result by Markus and Matsaev [17], [16, Theorem 6.12].

In Theorem 6.2 we obtain a Riesz basis with parentheses under a spectral condition of different type: we impose no restriction on the multiplicities and instead assume that the eigenvalues of G lie on sufficiently separated line segments, see Figure 4. If we know a priori that the eigenvalues of the perturbed operator T are

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uniformly separated, then Theorem 6.2 even yields a Riesz basis without parentheses. An example for such a situation may be found in Theorem 7.2. In contrast to our result, classical perturbation theorems for Riesz bases without parentheses such as Kato [11, Theorem V.4.15a], Dunford and Schwartz [7, Theorem XIX.2.7] and Clark [4] require that almost all eigenvalues of G are simple.

Apart from the above mentioned theorems, a wide range of existence results for Riesz bases of root vectors may be found in the literature, both for abstract operator settings and for concrete applications. Dissipative operators, for example, were considered by several authors; references and some results may be found in [9]. For generators of C_0 -semigroups, a Riesz basis of eigenvectors implies the so-called spectrum determined growth assumption, see [5, Theorem 2.3.5]. Zwart [25] obtained Riesz bases for generators of C_0 -groups, while Xu and Yung [24] constructed Riesz bases with parentheses for semigroup generators. Riesz basis properties of root vectors are also investigated for operator pencils, see e.g. [2, 22]. Pencils coming from concrete physical problems were studied in [1, 18].

Finally there are simple examples of non-normal operators whose eigenvectors are complete but do not form a Riesz basis, see e.g. [6].

In this paper, we follow ideas due to Markus and Matsaev [16, Chapter 1] to prove the existence of Riesz bases of root vectors. In Section 2 we start by deriving a completeness theorem for the system of root vectors of an operator with compact resolvent. Unlike the classical Keldysh theorem on completeness [12], [16, §4], where the resolvent belongs to a von Neumann-Schatten class, we assume here that it is uniformly bounded on an appropriate sequence of curves.

In Section 3 we then recall the notion of a Riesz basis consisting of subspaces and provide a sufficient condition for its existence in terms of projections. Although a Riesz basis consisting of (finite-dimensional) subspaces is equivalent to a Riesz basis with parentheses, we use the basis of subspaces notion in the formulation of our theorems, since it is more convenient.

In Section 4 we study in detail the change of the spectrum of a normal operator under a p -subordinate perturbation. The basic observation here is that if the spectrum of G lies on rays from the origin, then the spectrum of T lies inside parabolas around these rays. Based on the localisation of the spectrum, several estimates for Riesz projections of T are obtained in Section 5.

In Section 6 we derive our main existence results for Riesz bases of root vectors. In fact, these results also hold in the more general setting where G is a possibly non-normal operator with compact resolvent, a Riesz basis of root vectors and an appropriate spectrum, see Proposition 6.6 and Remark 6.7.

In Section 7 we finally apply our theory to diagonally dominant block operator matrices. Riesz bases of root vectors were obtained by Jacob, Trunk and Winklmeier [10] for operator matrices associated with damped vibrations of thin beams and by Kuiper and Zwart [14] for a class of Hamiltonian operator matrices from control theory. In Theorem 7.1 we consider operator matrices whose entries may all be unbounded, whereas in [10, 14] some of the entries were always bounded. Theorem 7.2 applies to a class of Hamiltonians which is different from the one in [14]. While the eigenvalues of the diagonal part of the Hamiltonian in [14] are simple, we consider the case of double eigenvalues.

2 Completeness of the system of root subspaces

We derive a completeness theorem for the system of root subspaces of an operator with compact resolvent, which applies to a different situation than the classical theorem of Keldysh [12], [16, §4].

Let T be an operator on a Banach space with a compact isolated part $\sigma \subset \sigma(T)$ of the spectrum. Let Γ be a simply closed, positively oriented integration contour with σ in its interior and $\sigma(T) \setminus \sigma$ in its exterior. Then

$$P = \frac{i}{2\pi} \int_{\Gamma} (T - z)^{-1} dz \quad (1)$$

defines a projection such that $\mathcal{R}(P)$ and $\ker P$ are T -invariant, $\mathcal{R}(P) \subset \mathcal{D}(T)$, and

$$\sigma(T|_{\mathcal{R}(P)}) = \sigma, \quad \sigma(T|_{\ker P}) = \sigma(T) \setminus \sigma.$$

P does not depend on the particular choice of Γ and is called the *Riesz projection* associated with σ (or Γ); $\mathcal{R}(P)$ is the corresponding *spectral subspace*. For a proof see [8, Theorem XV.2.1] or [11, Theorem III.6.17].

For an eigenvalue λ of T we call

$$\mathcal{L}(\lambda) = \bigcup_{k \in \mathbb{N}} \ker(T - \lambda)^k$$

the *root subspace* of T corresponding to λ ; the non-zero elements of $\mathcal{L}(\lambda)$ are the *root vectors*. A sequence of root vectors $x_1, \dots, x_n \in \mathcal{L}(\lambda)$ is called a *Jordan chain* if $(T - \lambda)x_k = x_{k-1}$ for $k \geq 2$ and $(T - \lambda)x_1 = 0$. In the case that T has a compact resolvent, its spectrum consists of isolated eigenvalues only; so for every eigenvalue λ there is the associated Riesz projection P_λ , which satisfies $\mathcal{R}(P_\lambda) = \mathcal{L}(\lambda)$.

Recall that for an operator T with compact resolvent on a Hilbert space its adjoint T^* also has a compact resolvent.

Lemma 2.1 *Let T be an operator with compact resolvent on a Hilbert space and M the subspace generated by all root subspaces of T , i.e., the set of all finite linear combinations of root vectors of T . If P is the Riesz projection of T^* corresponding to an eigenvalue $\lambda \in \sigma(T^*)$, then $M^\perp \subset \ker P$. Moreover, M^\perp is T^* -invariant and $(T^* - z)^{-1}$ -invariant for every $z \in \varrho(T^*)$; in particular $\varrho(T^*) \subset \varrho(T^*|_{M^\perp})$.*

Proof. We have $\lambda \in \sigma(T^*)$ if and only if $\bar{\lambda} \in \sigma(T)$. Observe that if P is the Riesz projection of T^* corresponding to λ , then P^* is the Riesz projection of T corresponding to $\bar{\lambda}$. Since $\mathcal{R}(P^*) \subset M$ we find $M^\perp \subset \mathcal{R}(P^*)^\perp = \ker P$. Now let $v \in M$ and $z \in \varrho(T^*)$. Then $Tv, (T - \bar{z})^{-1}v \in M$ and we have

$$\begin{aligned} u \in M^\perp \cap \mathcal{D}(T^*) &\Rightarrow (T^*u|v) = (u|Tv) = 0, \\ u \in M^\perp &\Rightarrow ((T^* - z)^{-1}u|v) = (u|(T - \bar{z})^{-1}v) = 0. \end{aligned}$$

Therefore M^\perp is T^* - and $(T^* - z)^{-1}$ -invariant, and this in turn implies the inclusion $\varrho(T^*) \subset \varrho(T^*|_{M^\perp})$. \square

Corollary 2.2 *Let T and M be as above. Then $\varrho(T^*|_{M^\perp}) = \mathbb{C}$.*

Proof. Since T has a compact resolvent, the same holds for T^* and $T^*|_{M^\perp}$. Consequently if $\lambda \in \sigma(T^*|_{M^\perp})$, then λ is an eigenvalue of $T^*|_{M^\perp}$, i.e., $T^*u = \lambda u$ for some $u \in M^\perp \setminus \{0\}$. In particular λ is an eigenvalue of T^* and we have $u \in \mathcal{R}(P)$ where P is the Riesz projection of T^* corresponding to λ . Now the previous lemma implies $u \in M^\perp \subset \ker P$ and hence $u = 0$, which is a contradiction. Therefore $\sigma(T^*|_{M^\perp}) = \emptyset$. \square

Theorem 2.3 *Let T be an operator with compact resolvent on a Hilbert space H with scalar product $(\cdot|\cdot)$. If there exists a sequence of bounded regions $(U_k)_{k \in \mathbb{N}}$ such that $\mathbb{C} = \bigcup_{k \in \mathbb{N}} U_k$, $\partial U_k \subset \varrho(T)$ for all k , and there is a constant $C \geq 0$ with*

$$\|(T - z)^{-1}\| \leq C \quad \text{for } z \in \partial U_k, k \in \mathbb{N},$$

*then the system of root subspaces of T is complete.*¹

Proof. Let M be as before. For $u, v \in M^\perp$ we consider the holomorphic function defined by

$$f(z) = ((T^*|_{M^\perp} - z)^{-1}u|v).$$

From the previous corollary we know that its domain of definition is \mathbb{C} . Since

$$\|(T^*|_{M^\perp} - z)^{-1}\| \leq \|(T^* - z)^{-1}\| = \|(T - \bar{z})^{-1}\| \quad \text{for } \bar{z} \in \varrho(T),$$

we see that $|f(z)| \leq C\|u\|\|v\|$ holds for $\bar{z} \in \partial U_k$. Using the maximum principle, we find that $|f(z)| \leq C\|u\|\|v\|$ for every $z \in \mathbb{C}$; by Liouville's theorem f is constant. Since u and v have been arbitrary, the mapping $z \mapsto (T^*|_{M^\perp} - z)^{-1}$ is also constant. For $u \in M^\perp$ we obtain

$$\begin{aligned} (T^*|_{M^\perp})^{-1}u &= (T^*|_{M^\perp} - I)^{-1}u \quad \Rightarrow \quad (T^*|_{M^\perp} - I)(T^*|_{M^\perp})^{-1}u = u \\ &\Rightarrow \quad (T^*|_{M^\perp})^{-1}u = 0 \quad \Rightarrow \quad u = 0. \end{aligned}$$

Hence $M^\perp = \{0\}$, i.e., $M \subset H$ is dense. \square

Corollary 2.4 *Let T be an operator with compact resolvent on a Hilbert space. Suppose that almost all eigenvalues of T lie in a finite number of pairwise disjoint sectors*

$$\Omega_j = \{z \in \mathbb{C} \mid |\arg z - \theta_j| < \psi_j\} \quad \text{with } 0 < \psi_j \leq \frac{\pi}{4}, \quad j = 1, \dots, n.$$

If there are constants $C, r_0 \geq 0$ such that

$$\|(T - z)^{-1}\| \leq C \quad \text{for } z \notin \Omega_1 \cup \dots \cup \Omega_n, |z| \geq r_0$$

and for each sector Ω_j there is a sequence $(x_k)_{k \in \mathbb{N}}$ with $x_k \rightarrow \infty$ and

$$\|(T - z)^{-1}\| \leq C \quad \text{for } z \in \Omega_j, \operatorname{Re}(e^{-i\theta_j}z) = x_k, k \in \mathbb{N},$$

then the system of root subspaces of T is complete. \square

¹A system of subspaces in H is called *complete* if the subspace generated by the system is dense in H .

3 Riesz bases of subspaces

We recall the closely related concepts of Riesz bases, Riesz bases with parentheses, and Riesz bases of subspaces, see [22, §1], [9, Chapter VI], [19, §15] and [23, §2] for more details.

Definition 3.1 Let H be a separable Hilbert space.

- (i) A sequence $(v_k)_{k \in \mathbb{N}}$ in H is called a *Riesz basis* of H if there is an isomorphism $J : H \rightarrow H$ such that $(Jv_k)_{k \in \mathbb{N}}$ is an orthonormal basis of H .
- (ii) A sequence of closed subspaces $(V_k)_{k \in \mathbb{N}}$ of H is called a *Riesz basis of subspaces* of H if there is an isomorphism $J : H \rightarrow H$ such that $(J(V_k))_{k \in \mathbb{N}}$ is a complete system of pairwise orthogonal subspaces. J

Other notions for Riesz bases of subspaces are “unconditional basis of subspaces” [22] or “ l^2 -decomposition” [19].

The sequence $(v_k)_{k \in \mathbb{N}}$ is a Riesz basis if and only if $\inf \|v_k\| > 0$, $\sup \|v_k\| < \infty$, and every $x \in H$ has a unique representation

$$x = \sum_{k=0}^{\infty} \alpha_k v_k, \quad \alpha_k \in \mathbb{C},$$

where the convergence of the series is unconditional. There is a similar characterisation for Riesz bases of subspaces, see [23, §2.2] and [9, §VI.5] for a proof:

Proposition 3.2 *For a sequence $(V_k)_{k \in \mathbb{N}}$ of closed subspaces of H the following assertions are equivalent:*

- (i) $(V_k)_{k \in \mathbb{N}}$ is a Riesz basis of subspaces for H .
- (ii) The sequence $(V_k)_{k \in \mathbb{N}}$ is complete and there exists $c \geq 1$ such that

$$c^{-1} \sum_{k \in F} \|x_k\|^2 \leq \left\| \sum_{k \in F} x_k \right\|^2 \leq c \sum_{k \in F} \|x_k\|^2 \quad (2)$$

for all finite subsets $F \subset \mathbb{N}$ and $x_k \in V_k$.

- (iii) Every $x \in H$ has a unique representation $x = \sum_{k=0}^{\infty} x_k$ with $x_k \in V_k$, where the convergence of the series is unconditional. □

To refer to the constant in (2), we shall also speak of a *Riesz basis of subspaces with constant c* .

A sequence $(v_k)_{k \in \mathbb{N}}$ in a Hilbert space H is called a *Riesz basis with parentheses* if there exists a Riesz basis of subspaces $(V_k)_{k \in \mathbb{N}}$ of H and a subsequence $(n_k)_k$ of \mathbb{N} with $n_0 = 0$ such that $(v_{n_k}, \dots, v_{n_{k+1}-1})$ is a basis of V_k . In this case every $x \in H$ has a unique representation

$$x = \sum_{k=0}^{\infty} \left(\sum_{j=n_k}^{n_{k+1}-1} \alpha_j v_j \right), \quad \alpha_j \in \mathbb{C},$$

where the series over k converges unconditionally.

The definition of a Riesz basis of subspaces generalises naturally to a family of closed subspaces $(V_k)_{k \in \Lambda}$, where the index set Λ is either finite or countably infinite; Proposition 3.2 continuous to hold in this context. In particular, a finite family (V_1, \dots, V_n) of closed subspaces is a Riesz basis of H if and only if the subspaces form a direct sum $H = V_1 \oplus \dots \oplus V_n$. Despite this equivalence, the Riesz basis notion is convenient even for finite families to specify the constant c in (2). An example is the next lemma, which is used in the proof of Theorem 6.2 to show that the root subspaces of an operator form a Riesz basis.

Lemma 3.3 *Let $(W_k)_{k \in \Lambda}$ be a Riesz basis of subspaces of H with constant c_0 . Let $(V_{kj})_{j \in J_k}$ be Riesz bases of subspaces of W_k for all $k \in \Lambda$ with common constant c_1 . Then the family $(V_{kj})_{k \in \Lambda, j \in J_k}$ is a Riesz basis of subspaces of H with constant $c_0 c_1$.*

Proof. Since $(W_k)_{k \in \Lambda}$ is complete in H and $(V_{kj})_{j \in J_k}$ is complete in W_k for every $k \in \Lambda$, the family $(V_{kj})_{k \in \Lambda, j \in J_k}$ is complete in H . Consider $F \subset \Lambda$ finite, $F_k \subset J_k$ finite for each $k \in F$, and $x_{kj} \in V_{kj}$. Then, using (2), we obtain

$$\left\| \sum_{\substack{k \in F \\ j \in F_k}} x_{kj} \right\|^2 \leq c_0 \sum_{k \in F} \left\| \sum_{j \in F_k} x_{kj} \right\|^2 \leq c_0 \sum_{k \in F} c_1 \sum_{j \in F_k} \|x_{kj}\|^2 = c_0 c_1 \sum_{\substack{k \in F \\ j \in F_k}} \|x_{kj}\|^2$$

and similarly $\left\| \sum_{k \in F, j \in F_k} x_{kj} \right\|^2 \geq c_0^{-1} c_1^{-1} \sum_{k \in F, j \in F_k} \|x_{kj}\|^2$. \square

Note that the existence of the common constant c_1 is guaranteed if only finitely many J_k consist of more than one element.

Our next aim is to derive a sufficient condition for a sequence of projections to generate a Riesz basis of subspaces.

Lemma 3.4 *Let $(x_k)_{k \in \mathbb{N}}$ be a sequence in a Banach space. If there exists $C \geq 0$ such that for every reordering $\phi : \mathbb{N} \xrightarrow{\text{bij}} \mathbb{N}$ and every $n \in \mathbb{N}$ we have $\left\| \sum_{k=0}^n x_{\phi(k)} \right\| \leq C$, then*

$$\sup_{n \in \mathbb{N}, \varepsilon_k = \pm 1} \left\| \sum_{k=0}^n \varepsilon_k x_k \right\| \leq 2C.$$

Proof. Let $\varepsilon_0, \dots, \varepsilon_n \in \{-1, 1\}$ and consider reorderings ϕ_1 and ϕ_2 that move all $+1$ and all -1 in the sequence $(\varepsilon_0, \dots, \varepsilon_n)$, respectively, to its beginning. Then, with n_1, n_2 appropriate, we obtain

$$\left\| \sum_{k=0}^n \varepsilon_k x_k \right\| \leq \left\| \sum_{\substack{k=0 \\ \varepsilon_k = +1}}^n x_k \right\| + \left\| \sum_{\substack{k=0 \\ \varepsilon_k = -1}}^n x_k \right\| = \left\| \sum_{k=0}^{n_1} x_{\phi_1(k)} \right\| + \left\| \sum_{k=0}^{n_2} x_{\phi_2(k)} \right\| \leq 2C. \quad \square$$

Lemma 3.5 *Let H be a Hilbert space, $x_0, \dots, x_n \in H$, and*

$$E = \{(\varepsilon_0, \dots, \varepsilon_n) \mid \varepsilon_k = \pm 1\}.$$

Then

$$2^{n+1} \sum_{k=0}^n \|x_k\|^2 = \sum_{\varepsilon \in E} \|\varepsilon_0 x_0 + \dots + \varepsilon_n x_n\|^2.$$

Proof. We use induction on n . The statement is true for the case $n = 0$ since $2\|x_0\|^2 = \|x_0\|^2 + \|-x_0\|^2$. Now suppose the statement holds for some $n \geq 0$; let

$$\tilde{E} = \{(\varepsilon_0, \dots, \varepsilon_{n+1}) \mid \varepsilon_k = \pm 1\}$$

and write $x_\varepsilon = \varepsilon_0 x_0 + \dots + \varepsilon_n x_n$. Then

$$\begin{aligned} \sum_{\varepsilon \in \tilde{E}} \|\varepsilon_0 x_0 + \dots + \varepsilon_{n+1} x_{n+1}\|^2 &= \sum_{\varepsilon \in E} (\|x_\varepsilon + x_{n+1}\|^2 + \|x_\varepsilon - x_{n+1}\|^2) \\ &= \sum_{\varepsilon \in E} (2\|x_\varepsilon\|^2 + 2\|x_{n+1}\|^2) = 2 \sum_{\varepsilon \in E} \|x_\varepsilon\|^2 + 2 \cdot 2^{n+1} \|x_{n+1}\|^2 \\ &= 2^{n+2} \left(\sum_{k=0}^n \|x_k\|^2 + \|x_{n+1}\|^2 \right). \end{aligned}$$

□

Lemma 3.6 *Let P_0, \dots, P_n be projections in a Hilbert space H with $P_j P_k = 0$ for $j \neq k$. Then*

$$C^{-2} \sum_{k=0}^n \|P_k x\|^2 \leq \left\| \sum_{k=0}^n P_k x \right\|^2 \leq C^2 \sum_{k=0}^n \|P_k x\|^2 \quad \text{for all } x \in H$$

where $C = \max\{\|\sum_{k=0}^n \varepsilon_k P_k\| \mid \varepsilon_k = \pm 1\}$.

Proof. We write $x_k = P_k x$ and use the last lemma considering that $\varepsilon \in E$ for which $\|\varepsilon_0 x_0 + \dots + \varepsilon_n x_n\|$ becomes maximal. Then we obtain

$$\sum_{k=0}^n \|P_k x\|^2 \leq \|\varepsilon_0 x_0 + \dots + \varepsilon_n x_n\|^2 = \left\| \left(\sum_{k=0}^n \varepsilon_k P_k \right) \left(\sum_{k=0}^n x_k \right) \right\|^2 \leq C^2 \left\| \sum_{k=0}^n P_k x \right\|^2.$$

On the other hand, if we choose $\varepsilon \in E$ such that $\|\varepsilon_0 x_0 + \dots + \varepsilon_n x_n\|$ is minimal, we find

$$\begin{aligned} \left\| \sum_{k=0}^n P_k x \right\|^2 &= \left\| \left(\sum_{k=0}^n \varepsilon_k P_k \right) \left(\sum_{k=0}^n \varepsilon_k x_k \right) \right\|^2 \\ &\leq C^2 \|\varepsilon_0 x_0 + \dots + \varepsilon_n x_n\|^2 \leq C^2 \sum_{k=0}^n \|P_k x\|^2. \end{aligned}$$

□

The following statement is a slight modification of a result² in the book of Markus [16, Lemma 6.2].

Proposition 3.7 *Let H be a Hilbert space and $(P_k)_{k \in \mathbb{N}}$ a sequence of projections in H satisfying $P_j P_k = 0$ for $j \neq k$. Suppose that the family $(\mathcal{R}(P_k))_{k \in \mathbb{N}}$ is complete in H and that*

$$\sum_{k=0}^{\infty} |(P_k x | y)| \leq C \|x\| \|y\| \quad \text{for all } x, y \in H \quad (3)$$

with some constant $C \geq 0$. Then $(\mathcal{R}(P_k))_{k \in \mathbb{N}}$ is a Riesz basis of subspaces of H with constant $c = 4C^2$.

² Under the weaker assumption $\sum_{k=0}^{\infty} |(P_k x | y)| < \infty$ for all $x, y \in H$, the existence of the Riesz basis of subspaces is proved, but without obtaining an estimate for the constant c .

Proof. From

$$\left| \left(\sum_{k=0}^n P_k x \middle| y \right) \right| \leq \sum_{k=0}^n |(P_k x | y)| \leq C \|x\| \|y\|$$

we conclude that $\|\sum_{k=0}^n P_k\| \leq C$ for all $n \in \mathbb{N}$. This assertion remains valid after an arbitrary rearrangement of the sequence $(P_k)_{k \in \mathbb{N}}$ since (3) still holds for the rearranged sequence. An application of Lemmas 3.4, 3.6 and Proposition 3.2 now completes the proof. \square

We end this section with a remark on the connection between Riesz bases of finite-dimensional invariant subspaces of an operator and Riesz bases with parentheses of root vectors, see also [23, §2.3].

Remark 3.8 Let T be an operator on a Hilbert space. Since every finite-dimensional T -invariant subspace³ admits a basis consisting of Jordan chains, it is immediate that a Riesz basis of finite-dimensional T -invariant subspaces is equivalent to a Riesz basis with parentheses of Jordan chains such that each Jordan chain lies inside some parenthesis. As a consequence of Lemma 3.3, a Riesz basis of finite-dimensional invariant subspaces where almost all subspaces are one-dimensional is equivalent to a Riesz basis of eigenvectors and finitely many Jordan chains.

4 Spectral enclosures for p -subordinate perturbations

The concept of p -subordination is in a certain sense an interpolation between the notions of boundedness and relative boundedness. We start with a result for relatively bounded perturbations.

Lemma 4.1 *Let G and S be operators on a Banach space with $\mathcal{D}(G) \subset \mathcal{D}(S)$ and $T = G + S$. If $0 < \varepsilon < 1$ and $z \in \varrho(G)$ such that*

$$\|S(G - z)^{-1}\| \leq \varepsilon, \tag{4}$$

then $z \in \varrho(T)$ and

$$\|(T - z)^{-1}\| \leq \frac{1}{1 - \varepsilon} \|(G - z)^{-1}\|, \quad \|S(T - z)^{-1}\| \leq \frac{\varepsilon}{1 - \varepsilon}.$$

Moreover if $\Gamma \subset \varrho(G)$ is a simply closed, positively oriented integration contour and (4) holds for all $z \in \Gamma$, then $\Gamma \subset \varrho(T)$ and for the Riesz projections Q and P of G and T associated with Γ there are isomorphisms

$$\mathcal{R}(Q) \cong \mathcal{R}(P), \quad \ker Q \cong \ker P.$$

Proof. (4) implies the convergence of the Neumann series

$$(I + S(G - z)^{-1})^{-1} = \sum_{k=0}^{\infty} (-S(G - z)^{-1})^k$$

³ In general, a subspace U is called T -invariant if $T(U \cap \mathcal{D}(T)) \subset U$. If we speak of a finite-dimensional T -invariant subspace U , we additionally assume that $\dim U < \infty$ and $U \subset \mathcal{D}(T)$.

with

$$\|(I + S(G - z)^{-1})^{-1}\| \leq \frac{1}{1 - \|S(G - z)^{-1}\|} \leq \frac{1}{1 - \varepsilon}.$$

Since

$$T - z = (I + S(G - z)^{-1})(G - z),$$

we conclude that $z \in \varrho(T)$ with

$$\|(T - z)^{-1}\| \leq \|(G - z)^{-1}\| \|(I + S(G - z)^{-1})^{-1}\| \leq \frac{1}{1 - \varepsilon} \|(G - z)^{-1}\|.$$

The identity $S(T - z)^{-1} = S(G - z)^{-1}(I + S(G - z)^{-1})^{-1}$ yields $\|S(T - z)^{-1}\| \leq \varepsilon(1 - \varepsilon)^{-1}$.

To prove the assertion about the Riesz projections, consider the operators $T_r = G + rS$ for $r \in [0, 1]$. We have the power series expansion

$$(I + rS(G - z)^{-1})^{-1} = \sum_{k=0}^{\infty} r^k (-S(G - z)^{-1})^k, \quad r \in [0, 1],$$

which converges uniformly in $z \in \Gamma$. Consequently $\Gamma \subset \varrho(T_r)$, and

$$(T_r - z)^{-1} = (G - z)^{-1}(I + rS(G - z)^{-1})^{-1}$$

is continuous in r uniformly for $z \in \Gamma$. Hence the Riesz projections P_r of T_r associated with Γ also depend continuously on r . Now if $\|P_r - P_s\| < 1$, then there are isomorphisms

$$\mathcal{R}(P_r) \cong \mathcal{R}(P_s), \quad \ker P_r \cong \ker P_s,$$

see [11, §I.4.6]. Since r ranges over a compact interval, the proof is complete. \square

The concept of p -subordinate perturbations was studied by Krein [13, §I.7.1] and Markus [16, §5], see also [23, §3.2].

Definition 4.2 Let G, S be operators on some Banach space and $p \in [0, 1]$. Then S is said to be p -subordinate to G if $\mathcal{D}(G) \subset \mathcal{D}(S)$ and there exists $b \geq 0$ such that

$$\|Su\| \leq b\|u\|^{1-p}\|Gu\|^p \quad \text{for all } u \in \mathcal{D}(G). \quad (5)$$

In this case there is a minimal constant $b \geq 0$ such that (5) holds, which is called the p -subordination bound of S to G . \lrcorner

If S is p -subordinate to G with $p < 1$, then S is relatively bounded to G with relative bound 0; if also $0 \in \varrho(G)$ and $q > p$, then S is q -subordinate to G .

Remark 4.3 In the case that G and S are operators on a Hilbert space and that G is normal with compact resolvent and $0 \in \varrho(G)$, the following can be shown [16, §5]: If SG^{-p} is bounded with $0 \leq p \leq 1$, then S is p -subordinate to G . If S is p -subordinate to G with $0 \leq p < 1$, then SG^{-q} is bounded for all $q > p$; in particular, S is relatively compact to G . \lrcorner

Now we investigate how the spectrum of a normal operator G changes under a p -subordinate perturbation S with $p < 1$. We consider the case that $\sigma(G)$ lies on rays from the origin and denote sectors in the complex plane by

$$\Omega(\varphi_-, \varphi_+) = \{re^{i\varphi} \mid r \geq 0, \varphi_- < \varphi < \varphi_+\} \quad \text{and} \quad \Omega(\varphi) = \Omega(-\varphi, \varphi).$$

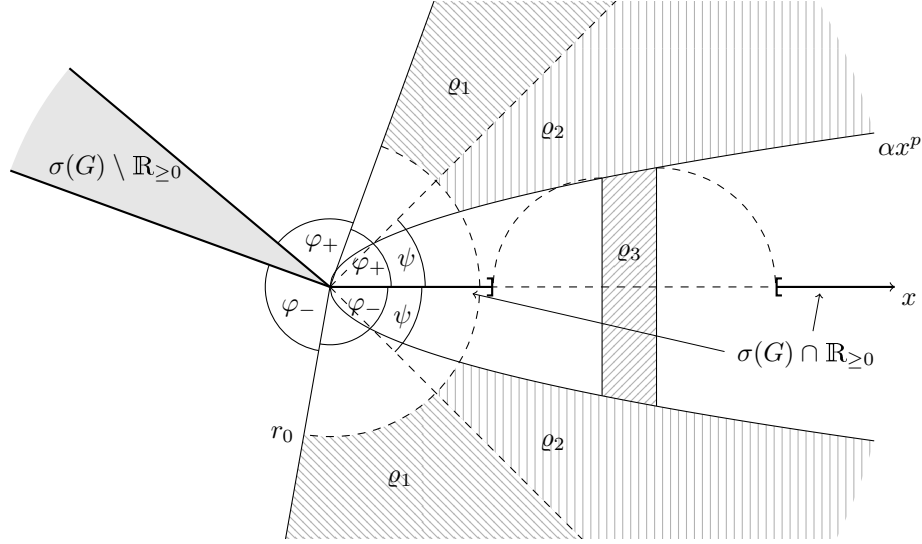


Figure 1: The situation of Lemma 4.4

In the next lemma the strip ϱ_3 corresponds to large gaps of $\sigma(G)$ on the positive real axis, compare Figure 1. Sufficient conditions for the existence of such gaps may be found in Proposition 5.8, Theorem 6.2 and Lemma 6.5.

Lemma 4.4 *Let G be a normal operator on a Hilbert space H such that $\sigma(G) \cap \Omega(2\varphi_-, 2\varphi_+) \subset \mathbb{R}_{\geq 0}$ with $-\pi \leq \varphi_- < 0 < \varphi_+ \leq \pi$. Let S be p -subordinate to G with bound b , $0 \leq p < 1$, and $T = G + S$.*

Then for $\alpha > b$, $b/\alpha < \varepsilon < 1$, and $0 < \psi < \min\{-\varphi_-, \varphi_+, \pi/2\}$ there exists $r_0 > 0$ such that the sets

$$\begin{aligned} \varrho_1 &= \{z \in \overline{\Omega(\varphi_-, \varphi_+)} \mid |z| \geq r_0, z \notin \Omega(\psi)\}, \\ \varrho_2 &= \{z = x + iy \in \overline{\Omega(\psi)} \mid |z| \geq r_0, |y| \geq \alpha x^p\}, \\ \varrho_3 &= \{z = x + iy \in \overline{\Omega(\psi)} \mid |z| \geq r_0, |y| \leq \alpha x^p \leq \text{dist}(z, \sigma(G))\} \end{aligned}$$

satisfy $\varrho_1 \cup \varrho_2 \cup \varrho_3 \subset \varrho(T)$, and for $z \in \varrho_1 \cup \varrho_2 \cup \varrho_3$ we have

$$\|S(G - z)^{-1}\| \leq \varepsilon, \quad \|(T - z)^{-1}\| \leq \frac{(1 - \varepsilon)^{-1}}{\text{dist}(z, \sigma(G))}, \quad \|S(T - z)^{-1}\| \leq \frac{\varepsilon}{1 - \varepsilon}.$$

Furthermore there is a constant $M > 0$ such that

$$\|(T - z)^{-1}\| \leq M \quad \text{for all } z \in \varrho_1 \cup \varrho_2 \cup \varrho_3.$$

Proof. We write $d = \text{dist}(z, \sigma(G))$ and use a consequence of the spectral theorem for normal operators, see [11, §V.3.8]:

$$\|(G - z)^{-1}\| = \sup_{\lambda \in \sigma(G)} \frac{1}{|\lambda - z|} = \frac{1}{d}, \quad \|G(G - z)^{-1}\| = \|I + z(G - z)^{-1}\| \leq 1 + \frac{|z|}{d}.$$

With the definition of p -subordination this yields

$$\|S(G - z)^{-1}u\| \leq b\|G(G - z)^{-1}u\|^p\|(G - z)^{-1}u\|^{1-p} \leq b\left(1 + \frac{|z|}{d}\right)^p \frac{1}{d^{1-p}}\|u\|$$

for every $u \in H$. In order to apply Lemma 4.1, we thus have to show that

$$C = b\left(1 + \frac{|z|}{d}\right)^p \frac{1}{d^{1-p}} \leq \varepsilon. \quad (6)$$

First we analyse the geometry of the situation: For $z = x + iy$ we have the implications

$$\varphi_- \leq \arg z \leq -\frac{\pi}{2} \text{ or } \frac{\pi}{2} \leq \arg z \leq \varphi_+ \implies d \geq |z|, \quad (7)$$

$$\max\left\{\varphi_-, -\frac{\pi}{2}\right\} \leq \arg z \leq \min\left\{\varphi_+, \frac{\pi}{2}\right\} \implies d \geq |y|, \quad (8)$$

as well as

$$\psi \leq |\arg z| \leq \frac{\pi}{2} \implies |y| \geq |z| \sin \psi, \quad (9)$$

$$|\arg z| \leq \psi \implies x \geq |z| \cos \psi. \quad (10)$$

Now let $z \in \varrho_1$. If $\varphi_- \leq \arg z \leq -\pi/2$ or $\pi/2 \leq \arg z \leq \varphi_+$, then (7) yields $C \leq 2^p b|z|^{p-1} \leq \varepsilon$, provided r_0 is large enough. If $\psi \leq |\arg z| \leq \pi/2$, then (8) and (9) imply $d \geq |z| \sin \psi$ and hence

$$C \leq b\left(1 + \frac{1}{\sin \psi}\right)^p \frac{1}{(|z| \sin \psi)^{1-p}} \leq \varepsilon$$

for r_0 sufficiently large.

For $z \in \varrho_2$, the implications (8) and (10) apply and with $|y| \geq \alpha x^p$ we find $d \geq \alpha x^p$. For $p > 0$ we use the Minkowski inequality to get the estimate

$$\left(1 + \frac{|z|}{d}\right)^p \leq \left(1 + \frac{x + |y|}{d}\right)^p \leq 1 + \frac{x^p + |y|^p}{d^p} \leq 1 + \frac{\alpha^{-1}d + d^p}{d^p} = 2 + \frac{1}{\alpha}d^{1-p},$$

i.e. $C \leq 2bd^{p-1} + b/\alpha$. Since $b/\alpha < \varepsilon$ and $d \geq \alpha(|z| \cos \psi)^p$, we obtain $C \leq \varepsilon$ for r_0 sufficiently large. On the other hand, if $p = 0$ then $d \geq \alpha$ and $C = b/d \leq b/\alpha < \varepsilon$.

In the case $z \in \varrho_3$, (8) and (10) apply, and we have $d \geq \alpha x^p$ by definition of the set ϱ_3 . In the same manner as for $z \in \varrho_2$, we conclude that $C \leq \varepsilon$ if r_0 is large enough.

Finally, to prove that $\|(T - z)^{-1}\|$ is uniformly bounded, we need to show that d^{-1} is bounded independently of z . For $z \in \varrho_1$ we have

$$\text{either } d \geq |z| \geq r_0 > 0 \text{ or } d \geq |z| \sin \psi \geq r_0 \sin \psi > 0.$$

For $z \in \varrho_2 \cup \varrho_3$ we obtain

$$d \geq \alpha(|z| \cos \psi)^p \geq \alpha(r_0 \cos \psi)^p > 0.$$

□

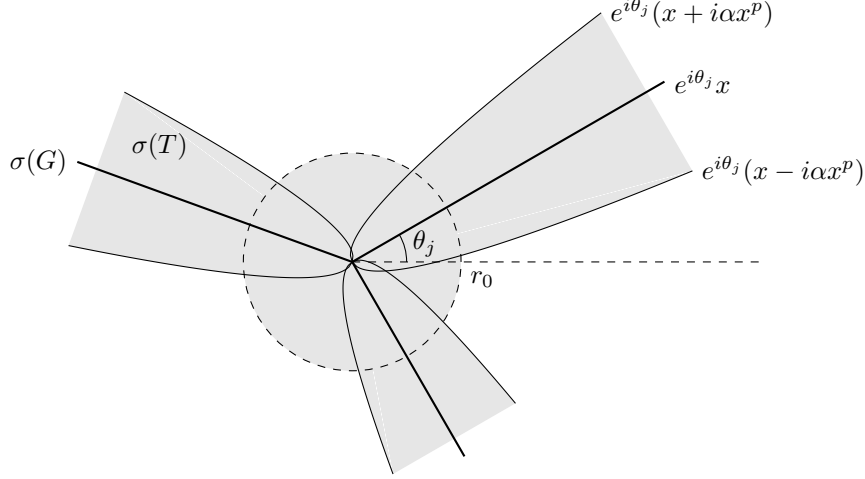


Figure 2: The spectrum after a p -subordinate perturbation

Theorem 4.5 *Let G be a normal operator whose spectrum lies on finitely many rays $e^{i\theta_j} \mathbb{R}_{\geq 0}$ with $0 \leq \theta_j < 2\pi$, $j = 1, \dots, n$. Let $T = G + S$ where S is p -subordinate to G with bound b and $0 \leq p < 1$. Then for every $\alpha > b$ there exists $r_0 > 0$ such that*

$$\sigma(T) \subset B_{r_0}(0) \cup \bigcup_{j=1}^n \{e^{i\theta_j}(x + iy) \mid x \geq 0, |y| \leq \alpha x^p\}, \quad (11)$$

cf. Figure 2. If G has a compact resolvent, then so has T .

Proof. Without loss of generality, we assume $\theta_1 < \theta_2 < \dots < \theta_n$ and set $\theta_0 = \theta_n - 2\pi$, $\theta_{n+1} = \theta_0 + 2\pi$. Then we may, after a rotation by θ_j , apply Lemma 4.4 to each sector $\Omega(\theta_{j-1}, \theta_{j+1})$. More precisely, we apply the lemma to the operators $e^{-i\theta_j}G$, $e^{-i\theta_j}S$, $e^{-i\theta_j}T$ with $\varphi_+ = (\theta_{j+1} - \theta_j)/2$, $\varphi_- = (\theta_{j-1} - \theta_j)/2$, and some suitable ε . This yields the implication

$$\begin{aligned} z \in \sigma(T), \quad \frac{\theta_{j-1} + \theta_j}{2} \leq \arg z \leq \frac{\theta_j + \theta_{j+1}}{2}, \quad |z| \geq r_0 \\ \implies z \in \{e^{i\theta_j}(x + iy) \mid x \geq 0, |y| \leq \alpha x^p\} \end{aligned}$$

with some $r_0 \geq 0$ for each $j = 1, \dots, n$. If G has compact resolvent, the identity

$$(T - z)^{-1} = (G - z)^{-1}(I + S(G - z)^{-1})^{-1} \quad \text{for } z \in \varrho(G) \cap \varrho(T)$$

implies that T has compact resolvent too. \square

The statement about the asymptotic shape of the spectrum of T can be refined as follows:

Remark 4.6 To obtain a condition for $z \in \varrho(T)$, we consider without loss of generality the case $\sigma(G) \cap \Omega(2\varphi) \subset \mathbb{R}_{\geq 0}$, $0 < \varphi \leq \pi/2$, and $z = x + iy \in \overline{\Omega(\varphi)}$.

Then $\text{dist}(z, \sigma(G)) \geq |y|$ and, in view of (6), $b(1 + |z|/|y|)^p |y|^{p-1} < 1$ is sufficient to get $z \in \varrho(T)$. For $p > 0$ this leads to the condition

$$x < \left(\frac{|y|}{b}\right)^{1/p} \sqrt{1 - 2b^{1/p}|y|^{1-1/p}},$$

which is asymptotically better than $x < (|y|/\alpha)^{1/p}$ from the theorem since $1 - 2b^{1/p}|y|^{1-1/p} \rightarrow 1$ as $|y| \rightarrow \infty$. For $p = 0$ we obtain the optimal condition $b < |y|$.

For $p > 0$, the estimates of Markus [16, Lemma 5.2] lead to asymptotics which are even slightly better. Also note that simply taking the limit $\alpha \rightarrow b$ in Theorem 4.5 is not possible since then also $r_0 \rightarrow \infty$. \square

5 Estimates for Riesz projections

In this section G is always a normal operator with compact resolvent on a Hilbert space H such that

$$\sigma(G) \cap \Omega(2\varphi) \subset \mathbb{R}_{\geq 0} \quad \text{with} \quad 0 < \varphi \leq \frac{\pi}{2}$$

and $T = G + S$ with S p -subordinate to G and $p < 1$.

The first two lemmas can be found in the book of Markus [16], for the special case $\alpha = 4b$. Since his proofs literally apply to the general situation, we omit them here; see also [23, §3.3].

Lemma 5.1 (Markus [16, Lemma 6.6]) *Let G be normal with compact resolvent and $\sigma(G) \cap \Omega(2\varphi) \subset \mathbb{R}_{\geq 0}$ with $0 < \varphi \leq \pi/2$. Then for $0 \leq p < 1$, $\alpha > 0$ there exists $r_0 > 0$ such that the contours*

$$\Gamma_{\pm} = \{x + iy \in \mathbb{C} \mid x \geq r_0, y = \pm \alpha x^p\} \quad (12)$$

satisfy $\Gamma_{\pm} \subset \varrho(G) \cap \overline{\Omega(\varphi)}$ and we have

$$\int_{\Gamma_{\pm}} |z|^p \|(G - z)^{-1}u\|^2 |dz| \leq C_1 \|u\|^2, \quad \int_{\Gamma_{\pm}} |z|^{p-2} \|G(G - z)^{-1}u\|^2 |dz| \leq C_2 \|u\|^2$$

for all $u \in H$ with some constants $C_1, C_2 \geq 0$. \square

Lemma 5.2 (Markus [16, Lemma 6.7]) *Let G be normal with compact resolvent and $\sigma(G) \cap \Omega(2\varphi) \subset \mathbb{R}_{\geq 0}$ with $0 < \varphi \leq \pi/2$. Let $(x_k)_{k \geq 1}$ be a sequence of positive numbers, $0 \leq p < 1$, and $\alpha, c_1, c_2 > 0$ such that $\alpha x_1^{p-1} \leq \tan \varphi$ and*

$$x_n^{1-p} - x_k^{1-p} \geq c_1(n - k) \quad \text{for } n > k, \quad \text{dist}(x_k, \sigma(G)) \geq c_2 x_k^p \quad \text{for } k \geq 1.$$

Then the lines

$$\gamma_k = \{x_k + iy \in \mathbb{C} \mid |y| \leq \alpha x_k^p\} \quad (13)$$

satisfy $\gamma_k \subset \varrho(G) \cap \overline{\Omega(\varphi)}$ and we have

$$\sum_{k=1}^{\infty} x_k^p \int_{\gamma_k} \|(G - z)^{-1}u\|^2 |dz| \leq C_1 \|u\|^2, \quad \sum_{k=1}^{\infty} x_k^{p-2} \int_{\gamma_k} \|G(G - z)^{-1}u\|^2 |dz| \leq C_2 \|u\|^2$$

for all $u \in H$ with some constants $C_1, C_2 \geq 0$. \square

With the previous resolvent estimates at hand, we derive an estimate for a sequence of Riesz projections associated with the parabola Γ_{\pm} and the lines γ_k :

Lemma 5.3 *Let G be normal with compact resolvent, $\sigma(G) \cap \Omega(2\varphi) \subset \mathbb{R}_{\geq 0}$ with $0 < \varphi \leq \pi/2$, S p -subordinate to G with bound b , $0 \leq p < 1$, and $T = G + \bar{S}$.*

Let $\alpha > b$, let $(x_k)_{k \geq 1}$, γ_k be as in Lemma 5.2, and suppose that there is a constant $M \geq 0$ such that

$$\gamma_k \subset \varrho(T) \quad \text{and} \quad \|S(T - z)^{-1}\| \leq M \quad \text{for all } z \in \gamma_k, k \geq 1.$$

Then there exist $r_0 > 0$, $k_0 \geq 1$ such that $x_{k_0} \geq r_0$ and the following holds: If Γ_{\pm} is as in (12) and Γ_k with $k \geq k_0$ is the positively oriented boundary contour of the region enclosed by $\gamma_k, \Gamma_-, \gamma_{k+1}, \Gamma_+$, then $\Gamma_k \subset \varrho(T)$. If P_k is the Riesz projection of T associated with Γ_k , then

$$\sum_{k=k_0}^{\infty} |(P_k u | v)| \leq C \|u\| \|v\| \quad \text{for all } u, v \in H \quad (14)$$

with some constant $C \geq 0$.

Proof. We want to apply Lemmas 4.4, 5.1 and 5.2, and choose $\varepsilon \in]b/\alpha, 1[$ and r_0 accordingly. The assumptions on the sequence $(x_k)_k$ imply that it tends monotonically to infinity and we choose k_0 such that $x_{k_0} \geq r_0$. By Lemma 4.4, $\|S(T - z)^{-1}\|$ is uniformly bounded on Γ_{\pm} . We thus have

$$\Gamma_k \subset \varrho(G) \cap \varrho(T) \quad \text{and} \quad \|S(T - z)^{-1}\| \leq M_0 \quad \text{for all } z \in \Gamma_k, k \geq k_0,$$

with some $M_0 \geq 0$. Consider now the Riesz projections Q_k of G associated with Γ_k , which are orthogonal since G is normal. It is easy to see that, to prove (14), it suffices to prove

$$\sum_{k=k_0}^{\infty} |((P_k - Q_k)u | v)| \leq C \|u\| \|v\|.$$

Now

$$P_k - Q_k = \frac{i}{2\pi} \int_{\Gamma_k} ((T - z)^{-1} - (G - z)^{-1}) dz = \frac{-i}{2\pi} \int_{\Gamma_k} (T - z)^{-1} S(G - z)^{-1} dz$$

and hence

$$|((P_k - Q_k)u | v)| \leq \frac{1}{2\pi} \int_{\Gamma_k} \|S(G - z)^{-1}u\| \|(T - z)^{-*}v\| |dz|.$$

Then, with the help of

$$\begin{aligned} (T - z)^{-1} &= (G - z)^{-1} (I - S(T - z)^{-1}) \\ \implies \|(T - z)^{-*}v\| &\leq (1 + \underbrace{\|S(T - z)^{-1}\|}_{\leq M_0}) \|(G - z)^{-*}v\| \end{aligned}$$

and $\|(G - z)^{-*}v\| = \|(G - z)^{-1}v\|$ (since G is normal), we find

$$\begin{aligned} \sum_{k=k_0}^{\infty} |((P_k - Q_k)u | v)| &\leq \frac{1 + M_0}{2\pi} \sum_{k=k_0}^{\infty} \int_{\Gamma_k} \|S(G - z)^{-1}u\| \|(G - z)^{-1}v\| |dz| \\ &\leq \frac{1 + M_0}{2\pi} \left(\int_{\Gamma_+} + \int_{\Gamma_-} + 2 \sum_{k=k_0}^{\infty} \int_{\gamma_k} \right) \|S(G - z)^{-1}u\| \|(G - z)^{-1}v\| |dz|. \end{aligned}$$

Using p -subordination, Lemma 5.1, and (for $p \neq 0$) Hölder's inequality, we estimate

$$\begin{aligned} & \int_{\Gamma_{\pm}} \|S(G-z)^{-1}u\| \|(G-z)^{-1}v\| |dz| \\ & \leq \left(\int_{\Gamma_{\pm}} |z|^{-p} \|S(G-z)^{-1}u\|^2 |dz| \right)^{1/2} \underbrace{\left(\int_{\Gamma_{\pm}} |z|^p \|(G-z)^{-1}v\|^2 |dz| \right)^{1/2}}_{\leq C_1 \|v\|^2}, \end{aligned}$$

$$\begin{aligned} & \int_{\Gamma_{\pm}} |z|^{-p} \|S(G-z)^{-1}u\|^2 |dz| \\ & \leq b^2 \left(\int_{\Gamma_{\pm}} |z|^{p-2} \|G(G-z)^{-1}u\|^2 |dz| \right)^p \left(\int_{\Gamma_{\pm}} |z|^p \|(G-z)^{-1}u\|^2 |dz| \right)^{1-p} \\ & \leq b^2 C_2^p C_1^{1-p} \|u\|^2. \end{aligned}$$

In the same way, with Lemma 5.2, we see that

$$\begin{aligned} & \sum_k \int_{\gamma_k} \|S(G-z)^{-1}u\| \|(G-z)^{-1}v\| |dz| \\ & \leq \left(\sum_k \int_{\gamma_k} x_k^{-p} \|S(G-z)^{-1}u\|^2 |dz| \right)^{1/2} \underbrace{\left(\sum_k \int_{\gamma_k} x_k^p \|(G-z)^{-1}v\|^2 |dz| \right)^{1/2}}_{\leq C'_1 \|v\|^2} \end{aligned}$$

and

$$\begin{aligned} & \sum_k \int_{\gamma_k} x_k^{-p} \|S(G-z)^{-1}u\|^2 |dz| \\ & \leq b^2 \left(\sum_k \int_{\gamma_k} x_k^{p-2} \|G(G-z)^{-1}u\|^2 |dz| \right)^p \left(\sum_k \int_{\gamma_k} x_k^p \|(G-z)^{-1}u\|^2 |dz| \right)^{1-p} \\ & \leq b^2 C_2'^p C_1'^{1-p} \|u\|^2. \end{aligned} \quad \square$$

To proceed, we need the concept of the determinant for operators, see [16, §2.5], [8, Chapter VII] and [9, §IV.1]. For an operator A of finite rank m , the *determinant* of $I + A$ is defined by

$$\det(I + A) = \det((I + A)|_{\mathcal{R}(A)}) \quad (15)$$

and it satisfies

$$(i) \quad |\det(I + A)| \leq (1 + \|A\|)^m;$$

(ii) $I + A$ is invertible if and only if $\det(I + A) \neq 0$, and in this case

$$\|(I + A)^{-1}\| \leq \frac{(1 + \|A\|)^m}{|\det(I + A)|};$$

(iii) if the operator-valued function $B : \Omega \rightarrow L(H)$ is analytic on a domain $\Omega \subset \mathbb{C}$, then $z \mapsto \det(I + AB(z))$ is analytic on Ω too.

We also use the following auxiliary result from complex analysis, cf. [16, Lemma 1.6], [15, Theorem I.11]:

Lemma 5.4 *Let $U \subset \mathbb{C}$ be a bounded, simply connected domain, $F \subset U$ compact, z_0 an interior point of F , and $\eta > 0$. Then there exists a constant $C > 0$ such that the following holds: If $a, b \in \mathbb{C}$ and $f : aU + b \rightarrow \mathbb{C}$ with $f(az_0 + b) \neq 0$ is holomorphic and bounded, then there is a set $E \subset \mathbb{C}$ being the union of finitely many discs with radii summing up to at most $|a|\eta$ such that*

$$|f(z)| \geq \frac{|f(az_0 + b)|^{1+C}}{\|f\|_{aU+b,\infty}^C} \quad \text{for all } z \in (aF + b) \setminus E.$$

The next proposition permits us to estimate the resolvent of the perturbed operator even close to its eigenvalues by artificially creating a gap in the spectrum of G . The method is taken from Lemma 5.6 in [16], which may be obtained from our proposition as a corollary. We denote by $N_+(r_1, r_2, G)$ the sum of the multiplicities of all the eigenvalues of G in the open interval $]r_1, r_2[$,

$$N_+(r_1, r_2, G) = \sum_{\lambda \in \sigma_p(G) \cap]r_1, r_2[} \dim \mathcal{L}(\lambda). \quad (16)$$

Proposition 5.5 *Let G be normal with compact resolvent, $\sigma(G) \cap \Omega(2\varphi) \subset \mathbb{R}_{\geq 0}$ with $0 < \varphi \leq \pi/2$, S p -subordinate to G with bound b , $0 \leq p < 1$, and $T = G + S$.*

Let $l > b$, $0 \leq l_0 < l - b$ and $\eta > 0$. Then there are constants $C_0, C_1, r_0 > 0$ such that for every $r \geq r_0$ there is a set $E_r \subset \mathbb{C}$ with the following properties:

(i) *E_r is the union of finitely many discs with radii summing up to at most ηr^p .*

(ii) *For every $z \in \overline{\Omega(\varphi)} \setminus E_r$ with $|\operatorname{Re} z - r| \leq l_0 r^p$ we have*

$$z \in \varrho(T) \quad \text{and} \quad \|(T - z)^{-1}\| \leq \frac{C_0 C_1^m}{r^p}, \quad \|S(T - z)^{-1}\| \leq C_0 C_1^m$$

where $m = N_+(r - lr^p, r + lr^p, G)$.

Proof. We choose $l_1 \in]l_0, l - b[$ and α, \tilde{b} such that

$$b < \tilde{b} < \alpha < l - l_1.$$

Let $r \geq r_0$. We may assume that $r - lr^p > 0$ by choosing r_0 large enough. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of G in $\Delta_r =]r - lr^p, r + lr^p[$, P_1, \dots, P_n the orthogonal projections onto the corresponding eigenspaces, and

$$K_r = \sum_{j=1}^n (\lambda_j - \tilde{\lambda}_j) P_j \quad \text{with} \quad \tilde{\lambda}_j = \begin{cases} r - lr^p & \text{if } \lambda_j < r, \\ r + lr^p & \text{if } \lambda_j \geq r. \end{cases}$$

Then $G_r = G - K_r$ is a normal operator with $\sigma(G_r) \cap \Omega(2\varphi) \subset \mathbb{R}_{\geq 0}$ and $\Delta_r \subset \varrho(G_r)$. K_r has rank m and satisfies $\|K_r\| \leq lr^p$. Noting that $\lambda_j / \tilde{\lambda}_j \leq r / (r - lr^p)$ for all j , it is straightforward to show that

$$\|Gu\| \leq \frac{r}{r - lr^p} \|G_r u\|.$$

Since $1 - lr^{p-1} \rightarrow 1$ as $r \rightarrow \infty$ and $b < \tilde{b}$, we conclude

$$\|Su\| \leq b\|Gu\|^p\|u\|^{1-p} \leq b\left(\frac{1}{1-lr^{p-1}}\right)^p\|G_ru\|^p\|u\|^{1-p} \leq \tilde{b}\|G_ru\|^p\|u\|^{1-p},$$

provided r_0 is sufficiently large. Thus S is p -subordinate to G_r with bound less or equal than \tilde{b} .

Next, we want to prove that

$$|x - r| \leq l_1 r^p \Rightarrow]x - \alpha x^p, x + \alpha x^p[\subset \varrho(G_r) \quad (17)$$

for r_0 sufficiently large. Let $|x - r| \leq l_1 r^p$. Since the function $x \mapsto x - \alpha x^p$ is monotonically increasing for large x , we have

$$x - \alpha x^p \geq r - l_1 r^p - \alpha(r - l_1 r^p)^p \geq r - l_1 r^p - \alpha r^p > r - lr^p$$

for r_0 large enough. Furthermore $\alpha(1 + l_1 r^{p-1})^p \leq l - l_1$ holds for large r and we obtain

$$x + \alpha x^p \leq r + l_1 r^p + \alpha(r + l_1 r^p)^p \leq r + lr^p.$$

In view of $\Delta_r \subset \varrho(G_r)$, (17) is proved.

Now we aim to apply Lemma 5.4 to the function

$$d(z) = \det(I + K_r(G_r + S - z)^{-1}), \quad z \in \varrho(G_r + S),$$

and the sets

$$\begin{aligned} U_r &= \{x + iy \mid |x - r| < l_1 r^p, |y| < 4br^p\}, \\ F_r &= \{x + iy \mid |x - r| \leq l_0 r^p, |y| \leq 3br^p\}. \end{aligned}$$

For r_0 sufficiently large we have $U_r \subset \Omega(\varphi)$. Using (17), we can apply Lemma 4.4 to $G_r + S$ with some $\varepsilon \in]\tilde{b}/\alpha, 1[$; we obtain $U_r \subset \varrho(G_r + S)$ and, for $z \in U_r$,

$$\text{dist}(z, \sigma(G_r)) \geq lr^p - l_1 r^p > \alpha r^p$$

and

$$\|(G_r + S - z)^{-1}\| \leq \frac{(1 - \varepsilon)^{-1}}{\alpha r^p}, \quad \|S(G_r + S - z)^{-1}\| \leq \frac{\varepsilon}{1 - \varepsilon}.$$

Then

$$|d(z)| \leq (1 + \|K_r\| \|(G_r + S - z)^{-1}\|)^m \leq \left(1 + \frac{l(1 - \varepsilon)^{-1}}{\alpha}\right)^m = c_0^m$$

on U_r with $c_0 > 0$. For $z \in \varrho(T) \cap U_r$ we have

$$I = (I + K_r(G_r + S - z)^{-1}) (I - K_r(T - z)^{-1}).$$

Applying Lemma 4.4 (now with $\varepsilon = 2/3$) to the operator T and $z_r = r + i \cdot 2br^p \in F_r$, we obtain

$$z_r \in \varrho(T) \quad \text{and} \quad \|(T - z_r)^{-1}\| \leq \frac{3}{2br^p}$$

and thus

$$\left| \frac{1}{d(z_r)} \right| = |\det(I - K_r(T - z_r)^{-1})| \leq \left(1 + \frac{3l}{2b}\right)^m = c_1^m$$

with $c_1 > 0$. Lemma 5.4 then yields a constant $C > 0$ depending only on b, l_0, l_1 and η such that for every $r \geq r_0$ there exists a union E_r of discs with radii summing up to at most ηr^p and

$$|d(z)| \geq c_0^{-mC} c_1^{-m(1+C)} \quad \text{for all } z \in F_r \setminus E_r.$$

For $z \in F_r \setminus E_r$, we thus obtain that $I + K_r(G_r + S - z)^{-1}$ is invertible with

$$\|(I + K_r(G_r + S - z)^{-1})^{-1}\| \leq \frac{c_0^m}{|d(z)|} \leq (c_0 c_1)^{(1+C)m}.$$

Consequently $z \in \varrho(T)$ with

$$(T - z)^{-1} = (G_r + S - z)^{-1} (I + K_r(G_r + S - z)^{-1})^{-1}$$

and

$$\begin{aligned} \|(T - z)^{-1}\| &\leq \frac{(1 - \varepsilon)^{-1}}{\alpha r^p} (c_0 c_1)^{(1+C)m} \leq \frac{C_0 C_1^m}{r^p}, \\ \|S(T - z)^{-1}\| &\leq \frac{\varepsilon}{1 - \varepsilon} (c_0 c_1)^{(1+C)m} \leq C_0 C_1^m \end{aligned}$$

with appropriate constants C_0, C_1 depending on $b, l, l_0, l_1, \eta, \alpha, \varepsilon$ only.

Finally, we consider $z = x + iy \in \overline{\Omega(\varphi)}$ with $|x - r| \leq l_0 r^p$ and $|y| \geq 3br^p$. Then

$$2bx^p \leq 2b(r + l_0 r^p)^p \leq 3br^p \leq |y|$$

holds for r_0 sufficiently large. Applying Lemma 4.4 (again with $\varepsilon = 2/3$), we obtain $z \in \varrho(T)$ and

$$\|(T - z)^{-1}\| \leq \frac{3}{|y|} \leq \frac{1}{br^p} \leq \frac{C_0 C_1^m}{r^p}, \quad \|S(T - z)^{-1}\| \leq 2 \leq C_0 C_1^m$$

for $C_0 \geq \max\{2, b^{-1}\}$ and $C_1 \geq 1$. □

Corollary 5.6 *Let G be normal with compact resolvent, $\sigma(G) \cap \Omega(2\varphi) \subset \mathbb{R}_{\geq 0}$ with $0 < \varphi \leq \pi/2$, S p -subordinate to G with bound b , $0 \leq p < 1$, and $T = G + S$.*

Then for $l_0, q > 0$ there are constants $C_0, C_1, r_0 > 0$ such that for every $r \geq r_0$ the following holds: For every $z = x + iy$ with $|x - r| \leq l_0 r^p$, $|y| \leq 2bx^p$ there exists $q_1 \in]0, q[$ such that

$$|w - z| = q_1 r^p \implies w \in \varrho(T), \quad \|(T - w)^{-1}\| \leq \frac{C_0 C_1^m}{r^p},$$

where $m = N_+(r - lr^p, r + lr^p, G)$ with $l = b + 2(l_0 + q)$.

Proof. We use Proposition 5.5 with $l = b + 2(l_0 + q)$, $l_0 + q$ replacing l_0 , and $\eta = q/3$. For z as in the claim and $|w - z| \leq qr^p$ we have $|\arg w| \leq \varphi$ (for r_0 large enough) and $|\operatorname{Re} w - r| \leq (l_0 + q)r^p$. Now the sum of the diameters of the discs in E_r is at most $2\eta r^p < qr^p$. Hence there exists $q_1 \in]0, q[$ such that $w \notin E_r$ for $|w - z| = q_1 r^p$ and the claim is proved. □

Under certain assumptions on the distribution of the eigenvalues of G on the positive real axis, we now obtain a sequence of closed integration contours in $\varrho(T)$ of the form in Lemma 5.3 and estimates for the associated Riesz projections.

Proposition 5.7 *Let G be normal with compact resolvent, $\sigma(G) \cap \Omega(2\varphi) \subset \mathbb{R}_{\geq 0}$ with $0 < \varphi \leq \pi/2$, S p -subordinate to G with bound b , $0 \leq p < 1$, and $T = G + S$.*

Assume that there is a sequence $(r_k)_{k \geq 1}$ of positive numbers tending monotonically to infinity and some $l > b$, $m \in \mathbb{N}_{\geq 1}$ such that

$$N_+(r_k - lr_k^p, r_k + lr_k^p, G) \leq m \quad \text{for all } k \geq 1. \quad (18)$$

Then there are constants $C, r_0 > 0$, $\alpha > b$, and a sequence $(x_k)_{k \geq 1}$ in $\mathbb{R}_{\geq 0}$ tending monotonically to infinity such that the following holds:

- (i) $z \in \overline{\Omega(\varphi)}$ with $\operatorname{Re} z = x_k$ implies $z \in \varrho(T)$, $\|(T - z)^{-1}\| \leq C$.
- (ii) The contours Γ_{\pm}, γ_k from (12) and (13) satisfy $\Gamma_{\pm}, \gamma_k \subset \varrho(T)$.
- (iii) If P_k is the Riesz projection of T associated with the region enclosed by $\gamma_k, \Gamma_-, \gamma_{k+1}, \Gamma_+$, then

$$\sum_{k=1}^{\infty} |(P_k u | v)| \leq C \|u\| \|v\| \quad \text{for all } u, v \in H.$$

Proof. We apply Proposition 5.5 with $l_0 = (l - b)/2$ and $\eta = l_0/2$ to $r = r_k$, $k \geq k_0$, k_0 appropriate. Since the sum of the diameters of the discs in E_r is at most $l_0 r_k^p$ and the interval $[r_k - l_0 r_k^p, r_k + l_0 r_k^p]$ contains at most m eigenvalues of G , we can find an x_k such that

$$|x_k - r_k| \leq l_0 r_k^p, \quad \operatorname{dist}(x_k, \sigma(G)) \geq \frac{l_0}{3m} r_k^p,$$

and that $z \in \overline{\Omega(\varphi)}$ with $\operatorname{Re} z = x_k$ implies

$$z \in \varrho(T), \quad \|(T - z)^{-1}\| \leq \frac{C_0 C_1^m}{r_k^p}, \quad \|S(T - z)^{-1}\| \leq C_0 C_1^m.$$

Then $x_k/r_k \rightarrow 1$ as $k \rightarrow \infty$ and we obtain

$$\operatorname{dist}(x_k, \sigma(G)) \geq c_2 x_k^p \quad \text{for } k \geq k_0$$

with $c_2 > 0$ and k_0 appropriately chosen. Since $x_k \rightarrow \infty$, for every k_1 there exists $k_2 > k_1$ such that $x_{k_2}^{1-p} - x_{k_1}^{1-p} \geq 1$. Passing to an appropriate subsequence, we can thus assume that

$$x_{k+1}^{1-p} - x_k^{1-p} \geq 1 \quad \text{for all } k \in \mathbb{N},$$

which yields

$$x_n^{1-p} - x_k^{1-p} \geq n - k \quad \text{for } n > k.$$

Now an application of Lemma 5.3 with $\alpha = 2b$ and the sequence $(x_k)_{k \geq k_0}$, k_0 large enough, completes the proof. \square

If the spectrum of G has sufficiently large gaps on $\mathbb{R}_{\geq 0}$, then the spectrum of T has corresponding gaps (cf. Figure 3):

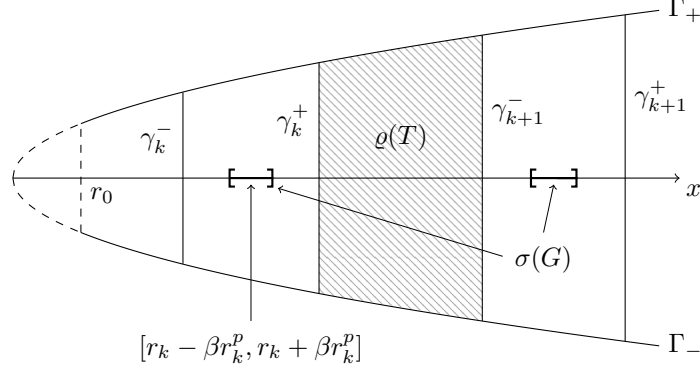


Figure 3: A large gap in $\sigma(G)$ yields a gap in $\sigma(T)$.

Proposition 5.8 *Let G be normal with compact resolvent, $\sigma(G) \cap \Omega(2\varphi) \subset \mathbb{R}_{\geq 0}$ with $0 < \varphi \leq \pi/2$, S p -subordinate to G with bound b , $0 \leq p < 1$, and $T = G + S$.*

Assume that there is a sequence $(r_k)_{k \geq 1}$ of nonnegative numbers tending monotonically to infinity and constants $\beta \geq 0$, $\alpha > b$, $l > \beta + \alpha$ such that

$$\sigma(G) \cap \mathbb{R}_{\geq 0} \subset \bigcup_{k \geq 1} [r_k - \beta r_k^p, r_k + \beta r_k^p] \quad (19)$$

and

$$r_k + l r_k^p \leq r_{k+1} - l r_{k+1}^p$$

for almost all k . Then there are constants $C, r_0 > 0$, $k_0 \geq 1$ such that the following holds:

(i) The contours Γ_{\pm} from (12) and

$$\gamma_k^{\pm} = \{x + iy \mid x = r_k \pm l r_k^p, |y| \leq \alpha x^p\} \quad \text{with } k \geq k_0$$

as well as the regions enclosed by γ_k^+ , γ_{k+1}^- , Γ_+ , Γ_- belong to $\varrho(T)$.

(ii) $z \in \overline{\Omega(\varphi)}$ with $\operatorname{Re} z = r_k + l r_k^p$, $k \geq k_0$, implies $\|(T - z)^{-1}\| \leq C$.

(iii) If P_k and Q_k are the Riesz projections of T and G , respectively, associated with the region enclosed by γ_k^- , γ_k^+ , Γ_+ , Γ_- , then

$$\sum_{k=k_0}^{\infty} |(P_k u|v)| \leq C \|u\| \|v\| \quad \text{for all } u, v \in H$$

and

$$\dim \mathcal{R}(P_k) = \dim \mathcal{R}(Q_k) \quad \text{for } k \geq k_0.$$

Proof. We set $s_k^{\pm} = r_k \pm l r_k^p$ so that $r_k \leq s_k^+ \leq s_{k+1}^- \leq r_{k+1}$. Consider $s \in [s_k^+, s_{k+1}^-]$ with $k \geq k_0$. Then

$$s + \alpha s^p \leq s_{k+1}^- + \alpha r_{k+1}^p = r_{k+1} - (l - \alpha) r_{k+1}^p \leq r_{k+1} - \beta r_{k+1}^p.$$

Furthermore we have

$$s - \alpha s^p \geq s_k^+ - \alpha(s_k^+)^p$$

for k_0 large enough, since the left-hand side is monotonically increasing in s for large s . In addition, the equivalent inequalities

$$s_k^+ - \alpha(s_k^+)^p \geq r_k + \beta r_k^p \Leftrightarrow l r_k^p - \alpha(r_k + l r_k^p)^p \geq \beta r_k^p$$

hold for k_0 sufficiently large since $1 + l r_k^{p-1} \rightarrow 1$. Using (19), we have thus proved that, for $k \geq k_0$,

$$s \in [s_k^+, s_{k+1}^-] \Rightarrow]s - \alpha s^p, s + \alpha s^p[\subset \varrho(G).$$

With r_0 and k_0 appropriately chosen, Lemma 4.4 implies that the region enclosed by γ_k^+ , γ_{k+1}^- , Γ^+ , and Γ^- as well as the contours itself belong to $\varrho(T)$ for $k \geq k_0$. Moreover, $\|(T - z)^{-1}\|$ and $\|S(T - z)^{-1}\|$ are uniformly bounded for $z \in \overline{\Omega(\varphi)}$ with $\operatorname{Re} z = s_k^+$, $k \geq k_0$. We also have $\operatorname{dist}(s_k^+, G) \geq \alpha(s_k^+)^p$ and

$$s_{k+1}^+ - s_k^+ = r_{k+1} - r_k + l(r_{k+1}^p - r_k^p) \geq 2l r_{k+1}^p.$$

The mean value theorem then yields

$$(s_{k+1}^+)^{1-p} - (s_k^+)^{1-p} \geq (1-p)(s_{k+1}^+)^{-p}(s_{k+1}^+ - s_k^+) \geq \frac{2l(1-p)r_{k+1}^p}{(r_{k+1} + l r_{k+1}^p)^p},$$

i.e., $(s_{k+1}^+)^{1-p} - (s_k^+)^{1-p} \geq l(1-p)$ for $k \geq k_0$, k_0 sufficiently large. We can thus apply Lemma 5.3 with $x_k = s_k^+$ to get the estimate for the sum over the Riesz projections. The final claim is a consequence of Lemmas 4.1 and 4.4. \square

6 Existence of Riesz bases of invariant subspaces

Let G be an operator with compact resolvent. Recall that we denote by $N_+(r_1, r_2, G)$ the sum of the multiplicities of the eigenvalues of G in the interval $]r_1, r_2[$, see (16). Similarly, we write

$$N(r, G) = \sum_{\lambda \in \sigma_p(G) \cap \overline{B_r(0)}} \dim \mathcal{L}(\lambda) \quad (20)$$

for the sum of the multiplicities of all the eigenvalues λ with $|\lambda| \leq r$ and

$$N(K, G) = \sum_{\lambda \in \sigma_p(G) \cap K} \dim \mathcal{L}(\lambda) \quad \text{for every set } K \subset \mathbb{C}. \quad (21)$$

Our first existence result for Riesz bases of invariant subspaces improves a theorem due to Markus and Matsaev [17], [16, Theorem 6.12]; there, condition (22) was formulated with \limsup instead of \liminf .

Theorem 6.1 *Let G be a normal operator with compact resolvent whose spectrum lies on a finite number of rays from the origin. Let S be p -subordinate to G with $0 \leq p < 1$. If*

$$\liminf_{r \rightarrow \infty} \frac{N(r, G)}{r^{1-p}} < \infty, \quad (22)$$

then $T = G + S$ admits a Riesz basis of finite-dimensional T -invariant subspaces.

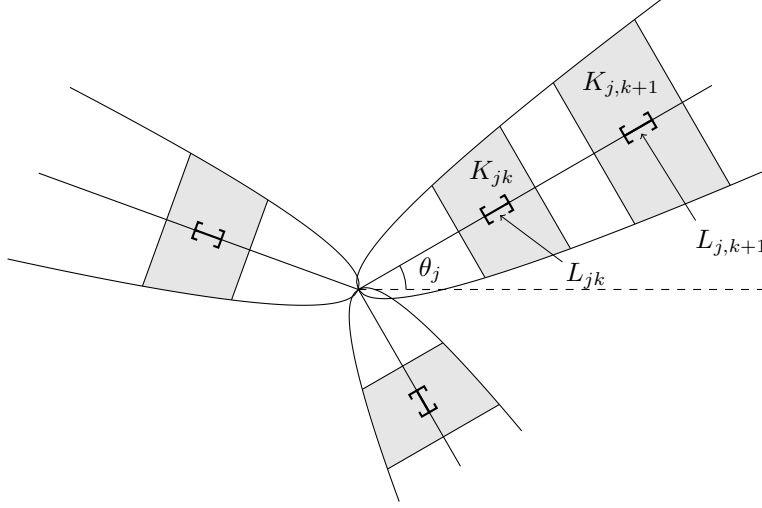


Figure 4: The situation of Theorem 6.2

Proof. Let $e^{i\theta_j}\mathbb{R}_{\geq 0}$ with $0 \leq \theta_1 < \dots < \theta_n < 2\pi$ be the rays containing the eigenvalues of G and let S be p -subordinate to G with bound b . From Theorem 4.5 we know that T has a compact resolvent and that almost all of its eigenvalues lie inside sectors of the form

$$\Omega_j = \{z \in \mathbb{C} \mid |\arg z - \theta_j| < \psi_j\} \quad \text{with} \quad 0 < \psi_j \leq \frac{\pi}{4},$$

where the ψ_j can be chosen such that these sectors are disjoint. Lemma 4.4 shows that $\|(T - z)^{-1}\|$ is uniformly bounded for $z \notin \Omega_1 \cup \dots \cup \Omega_n$, $|z| \geq r_0$. From the assumption (22) it can be shown that for each sector Ω_j there is a sequence $(r_{jk})_{k \geq 1}$ of positive numbers tending monotonically to infinity such that

$$\sup_k N_+(r_{jk} - 2br_{jk}^p, r_{jk} + 2br_{jk}^p, e^{-i\theta_j}G) < \infty,$$

see [16, Lemma 6.11]. By Proposition 5.7 we thus obtain a corresponding sequence $(x_{jk})_{k \geq 1}$ such that $\|(T - z)^{-1}\|$ is uniformly bounded for $z \in \Omega_j$, $\operatorname{Re}(e^{-i\theta_j}z) = x_{jk}$. Corollary 2.4 implies that the system of root subspaces of T is complete.

Furthermore, if $(P_{jk})_{k \geq 1}$ are the Riesz projections from Proposition 5.7 corresponding to the eigenvalues $\lambda \in \Omega_j$ of T with $\operatorname{Re}(e^{-i\theta_j}\lambda) > x_{j1}$ and P_0 is the Riesz projection for the (finitely many) remaining eigenvalues, then

$$|(P_0 u|v)| + \sum_{j=1}^n \sum_{k=1}^{\infty} |(P_{jk} u|v)| \leq C \|u\| \|v\|$$

with some constant $C \geq 0$. Now Proposition 3.7 shows that the ranges of the projections $P_0, (P_{jk})_{j,k}$ form a Riesz basis. \square

Replacing condition (22) by an assumption on the localisation of the spectrum of G on the rays, we obtain our second perturbation theorem.

Theorem 6.2 *Let G be a normal operator with compact resolvent on a Hilbert space H and S p -subordinate to G with bound b and $0 \leq p < 1$. Suppose that the spectrum of G lies on sequences of line segments on rays from the origin,*

$$\sigma(G) \subset \bigcup_{j=1}^n \bigcup_{k \geq 1} L_{jk}, \quad L_{jk} = \{e^{i\theta_j} x \mid x \geq 0, |x - r_{jk}| \leq \beta_j r_{jk}^p\}, \quad (23)$$

where $\beta_j \geq 0$, $0 \leq \theta_1 < \dots < \theta_n < 2\pi$, and $(r_{jk})_{k \geq 1}$ are monotonically increasing sequences of nonnegative numbers such that

$$r_{jk} + l_j r_{jk}^p \leq r_{j,k+1} - l_j r_{j,k+1}^p \quad (24)$$

for almost all k with some constants $l_j > \beta_j + b$.

Then $T = G + S$ has compact resolvent; for $b < \alpha < \min\{l_1 - \beta_1, \dots, l_n - \beta_n\}$ almost all eigenvalues of T lie inside the regions

$$K_{jk} = \{e^{i\theta_j}(x + iy) \mid x \geq 0, |x - r_{jk}| \leq (\beta_j + \alpha)r_{jk}^p, |y| \leq \alpha x^p\}, \quad (25)$$

$j = 1, \dots, n$, $k \geq 1$ (cf. Figure 4); the spectral subspaces of T corresponding to the regions K_{jk} together with the subspace corresponding to $\sigma(T) \setminus \bigcup_{j,k} K_{jk}$ form a Riesz basis of H ; and we have

$$N(L_{jk}, G) = N(K_{jk}, T) \quad \text{for almost all pairs } (j, k). \quad (26)$$

Moreover, if there are constants $m, q > 0$ such that for almost all pairs (j, k) the assertions

$$(i) \quad N(L_{jk}, G) \leq m \quad \text{and}$$

$$(ii) \quad \lambda_1, \lambda_2 \in \sigma(T) \cap K_{jk}, \lambda_1 \neq \lambda_2 \quad \Rightarrow \quad |\lambda_1 - \lambda_2| > q r_{jk}^p$$

hold, then the root subspaces of T form a Riesz basis of H .

Proof. We apply Theorem 4.5 and, for each ray, Proposition 5.8 with α and l replaced by $\tilde{\alpha} = (\alpha + b)/2$ and $\tilde{l}_j = \beta_j + \alpha$, respectively. This shows that T has a compact resolvent and that almost all of its eigenvalues lie inside regions

$$\{e^{i\theta_j}(x + iy) \mid x \geq 0, |x - r_{jk}| < \tilde{l}_j r_{jk}^p, |y| < \tilde{\alpha} x^p\} \subset K_{jk}.$$

By Lemma 4.4, $\|(T - z)^{-1}\|$ is uniformly bounded outside certain disjoint sectors Ω_j around the rays for $|z|$ large enough. For each ray, Proposition 5.8 yields a sequence $(x_{jk})_{k \in \mathbb{N}}$ tending monotonically to infinity such that $\|(T - z)^{-1}\|$ is bounded for $z \in \Omega_j$, $\operatorname{Re}(e^{-i\theta_j} z) = x_{jk}$. With Corollary 2.4 we conclude that the system of root subspaces of T is complete. Moreover, we have

$$|(P_0 u|v)| + \sum_{j=1}^n \sum_{k=1}^{\infty} |(P_{jk} u|v)| \leq C \|u\| \|v\|$$

for some $C \geq 0$ where P_{jk} is the Riesz projection associated with K_{jk} and P_0 the one associated with $\sigma(T) \setminus \bigcup_{j,k} K_{jk}$; Proposition 3.7 then yields the Riesz basis property. Finally, if Q_{jk} is the spectral projection of G associated with L_{jk} , Proposition 5.8 implies $\dim \mathcal{R}(Q_{jk}) = \dim \mathcal{R}(P_{jk})$ for almost all (j, k) and thus (26).

Now suppose that with $m, q > 0$ the additional assumptions (i) and (ii) hold for almost all pairs (j, k) . We aim to show that the root subspaces corresponding to the eigenvalues of T in each K_{jk} form a Riesz basis of $\mathcal{R}(P_{jk})$ with constant c independent of (j, k) . Without loss of generality we may assume

$$\theta_j = 0 \quad \text{and} \quad \beta_j + \alpha + q \leq l_j.$$

We want to apply Corollary 5.6 with $l_0 = \beta_j + \alpha$ and set l accordingly. From the relation $r_{j,k+1} - r_{jk} \geq 2l_j r_{jk}^p$ it is easy to verify that the number of elements r_{jk} in the interval $[r - lr^p, r + lr^p]$ is at most $2l/l_j$ for r sufficiently large. Hence there is a constant m_0 such that

$$N_+(r - lr^p, r + lr^p, G) \leq m_0 \quad \text{for } r \text{ sufficiently large.}$$

Let λ be an eigenvalue of T in K_{jk} . By Corollary 5.6 there exists $q_1 \in]0, q[$ such that the points w on the circle around λ with radius $q_1 r_{jk}^p$ satisfy $\|(T - w)^{-1}\| \leq C_0 C_1^{m_0} r_{jk}^{-p}$. In addition, this circle lies inside the strip $|\operatorname{Re} z - r_{jk}| \leq l_j r_{jk}^p$ and assumption (ii) thus implies that λ is the only possible eigenvalue of T inside that circle. Therefore, the Riesz projection P_λ for λ satisfies

$$\|P_\lambda\| \leq 2\pi q_1 r_{jk}^p \frac{C_0 C_1^{m_0}}{r_{jk}^p} \leq 2\pi q C_0 C_1^{m_0}.$$

If $\lambda_1, \dots, \lambda_{m_1}$ are the eigenvalues of T in K_{jk} , we have $m_1 \leq N(K_{jk}, T) \leq m$ and conclude

$$\sum_{s=1}^{m_1} |(P_{\lambda_s} u | v)| \leq 2\pi m q C_0 C_1^{m_0} \|u\| \|v\|.$$

According to Proposition 3.7, the subspaces $\mathcal{R}(P_{\lambda_s})$, $s = 1, \dots, m_1$, form a Riesz basis of $\mathcal{R}(P_{jk})$ with constant c independent of k . This is true for almost all pairs (j, k) , and hence an application of Lemma 3.3 shows that the root subspaces of T form a Riesz basis of H . \square

Remark 6.3 If almost all eigenvalues of G are simple and almost all line segments L_{jk} contain one eigenvalue only, then Theorem 6.2 yields a Riesz basis of eigenvectors and finitely many Jordan chains for T . Indeed almost all spectral subspaces corresponding to the K_{jk} are one-dimensional in this case, and the Riesz basis of subspaces is thus equivalent to the existence of a Riesz basis of eigenvectors and finitely many Jordan chains. \lrcorner

Remark 6.4 It can be shown [23, Lemma 3.4.9] that, if G satisfies the spectral condition (23) with some $\beta_j > 0$ such that $r_{jk} + \beta_j r_{jk}^p \leq r_{j,k+1} - \beta_j r_{j,k+1}^p$ and $N(L_{jk}, G)$ is bounded in (j, k) , then $\sup_{r \geq 1} N(r, G) r^{p-1} < \infty$; in particular the spectral condition (22) of Theorem 6.1 holds. However, the first part of Theorem 6.2 is applicable even if $N(L_{jk}, G)$ is unbounded and (22) does not hold. \lrcorner

The condition (24) can be reformulated for sequences with a certain asymptotic behaviour:

Lemma 6.5 Consider the sequence of nonnegative numbers given by

$$r_k = ck^q + d_k k^{q-1}$$

with $c > 0$, $q \geq 1$ and a converging sequence $(d_k)_{k \in \mathbb{N}}$. Then for $l, p \geq 0$ the relation

$$r_k + lr_k^p \leq r_{k+1} - lr_{k+1}^p$$

holds for almost all $k \in \mathbb{N}$ if

(i) $p < 1 - 1/q$, or

(ii) $p = 1 - 1/q$ and $l < qc^{1/q}/2$.

Proof. This can be shown in a straightforward way by a Taylor series expansion of r_{k+1} in k . \square

The next proposition reverses the assertions of the Theorems 4.5, 6.1 and 6.2 to some extent. As a consequence, the assumptions in these theorems can be relaxed.

Proposition 6.6 Let G be an operator on a Hilbert space H with compact resolvent and a Riesz basis of Jordan chains. Suppose that $0 \leq p < 1$, $\alpha \geq 0$, $0 \leq \theta_j < 2\pi$, $j = 1, \dots, n$, such that either

(i) there exists $r_0 > 0$ with

$$\sigma(G) \subset B_{r_0}(0) \cup \bigcup_{j=1}^n \{e^{i\theta_j}(x + iy) \mid x > 0, |y| \leq \alpha x^p\}, \quad \text{or}$$

(ii) almost all eigenvalues of G lie inside regions

$$K_{jk} = \{e^{i\theta_j}(x + iy) \mid r_{jk}^- \leq x \leq r_{jk}^+, |y| \leq \alpha x^p\}, \quad j = 1, \dots, n, k \geq 1,$$

where $(r_{jk}^\pm)_{k \geq 1}$ are sequences of positive numbers satisfying $r_{jk}^- \leq r_{jk}^+ < r_{j,k+1}^-$.

Then there is an isomorphism $J : H \rightarrow H$, a normal operator G_0 on H with compact resolvent, and an operator S_0 p -subordinate to G_0 such that

$$J\mathcal{D}(G) = \mathcal{D}(G_0), \quad JGJ^{-1} = G_0 + S_0.$$

In case (i), all eigenvalues of G_0 lie on the rays $e^{i\theta_j}\mathbb{R}_{\geq 0}$ and we have

$$N(r, G_0) = N(r, G) \quad \text{for } r \geq 1.$$

In case (ii), all eigenvalues of G_0 lie on the line segments

$$L_{jk} = \{e^{i\theta_j}x \mid r_{jk}^- \leq x \leq r_{jk}^+\},$$

and $N(L_{jk}, G_0) = N(K_{jk}, G)$ holds for almost all pairs (j, k) .

Moreover, if S is p -subordinate to G , then JSJ^{-1} is p -subordinate to G_0 .

Proof. The idea is to use the isomorphism J to transform the Riesz basis of Jordan chains of G to an orthonormal basis of eigenvectors of G_0 . Then one associates with each eigenvalue $\lambda = e^{i\theta_j}(x + iy)$ of G an eigenvalue $\mu = e^{i\theta_j}w$ of G_0 with $w > 0$ such that the assertions on the spectrum hold. A complete proof can be found in [23, §3.4]. \square

Remark 6.7 Theorems 4.5 and 6.1 also hold if G is as in the previous proposition and satisfies condition 6.6(i). Indeed we have

$$J(G + S)J^{-1} = G_0 + S_0 + JSJ^{-1}$$

in this case, $S_0 + JSJ^{-1}$ is p -subordinate to G_0 , and the theorems can be applied to the right-hand side. Analogously, Theorem 6.2 also holds if G satisfies 6.6(ii). In both cases, b is now the p -subordination bound of $S_0 + JSJ^{-1}$ to G_0 .

7 Application to block operator matrices

We apply Theorems 6.1 and 6.2 to two classes of diagonally dominant block operator matrices. For many results about the spectral theory of block operator matrices see [20, 21].

Theorem 7.1 *Let $A(H_1 \rightarrow H_1)$ and $D(H_2 \rightarrow H_2)$ be normal operators with compact resolvents on Hilbert spaces such that the spectra of A and D lie on finitely many rays from the origin and*

$$\liminf_{r \rightarrow \infty} \frac{N(r, A)}{r^{1-p}} < \infty, \quad \liminf_{r \rightarrow \infty} \frac{N(r, D)}{r^{1-p}} < \infty$$

with $0 \leq p < 1$. Suppose that the operators $C(H_1 \rightarrow H_2)$ and $B(H_2 \rightarrow H_1)$ are p -subordinate⁴ to A and D , respectively,

$$\begin{aligned} \|Cu\| &\leq b\|u\|^{1-p}\|Au\|^p \quad \text{for } u \in \mathcal{D}(A) \subset \mathcal{D}(C), \\ \|Bv\| &\leq b\|v\|^{1-p}\|Dv\|^p \quad \text{for } v \in \mathcal{D}(D) \subset \mathcal{D}(B). \end{aligned}$$

Then the block operator matrix

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

acting on $H_1 \times H_2$ has a compact resolvent, admits a Riesz basis of finite-dimensional T -invariant subspaces, and for every $\alpha > b$ there is a constant $r_0 \geq 0$ such that

$$\sigma(T) \subset B_{r_0}(0) \cup \bigcup_{j=1}^n \{e^{i\theta_j}(x + iy) \mid x \geq 0, |y| \leq \alpha x^p\}.$$

Here $\theta_1, \dots, \theta_n$ with $0 \leq \theta_j < 2\pi$ are the angles of the rays on which the spectra of A and D lie.

⁴ This notion of p -subordination is more general than the one from Section 4, since the operators B and C map from one Hilbert space into a (possibly) different one.

Proof. We apply Theorems 4.5 and 6.1 to the decomposition

$$T = G + S \quad \text{with} \quad G = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad S = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.$$

Indeed G is normal with compact resolvent and

$$\sigma(G) = \sigma(A) \cup \sigma(D), \quad N(r, G) = N(r, A) + N(r, D).$$

Moreover, using Hölder's inequality, we find

$$\begin{aligned} \left\| S \begin{pmatrix} u \\ v \end{pmatrix} \right\|^2 &= \|Bv\|^2 + \|Cu\|^2 \leq b^2 \|v\|^{2(1-p)} \|Dv\|^{2p} + b^2 \|u\|^{2(1-p)} \|Au\|^{2p} \\ &\leq b^2 (\|u\|^2 + \|v\|^2)^{1-p} (\|Au\|^2 + \|Dv\|^2)^p \end{aligned}$$

for $u \in \mathcal{D}(A)$, $v \in \mathcal{D}(D)$, i.e.

$$\|Sw\| \leq b \|w\|^{1-p} \|Gw\|^p \quad \text{for} \quad w \in \mathcal{D}(G) = \mathcal{D}(A) \times \mathcal{D}(D);$$

S is p -subordinate to G . □

In the next theorem, a symmetry of the operator matrix with respect to an indefinite inner product yields a gap in the spectrum around the imaginary axis. This makes it possible to apply the second part of Theorem 6.2.

Theorem 7.2 *Let A be a skew-adjoint operator with compact resolvent on a Hilbert space H . Let $B, C : H \rightarrow H$ be bounded, selfadjoint and uniformly positive, $B, C \geq \gamma > 0$. Write $(ir_k)_{k \in \Lambda}$ for the sequence of eigenvalues of A where $\Lambda \in \{\mathbb{Z}_+, \mathbb{Z}_-, \mathbb{Z}\}$ and $(r_k)_k$ is increasing. Suppose that almost all eigenvalues ir_k are simple and that for some $l > b = \max\{\|B\|, \|C\|\}$ we have*

$$r_{k+1} - r_k \geq 2l \quad \text{for almost all } k \in \Lambda.$$

Then the block operator matrix

$$T = \begin{pmatrix} A & B \\ C & A \end{pmatrix}$$

has a compact resolvent, its spectrum is symmetric with respect to the imaginary axis and satisfies

$$\sigma(T) \subset \{z \in \mathbb{C} \mid |z - ir_k| \leq b \text{ for some } k, |\operatorname{Re} z| \geq \gamma\}.$$

Moreover almost all eigenvalues are simple and T admits a Riesz basis of eigenvectors and finitely many Jordan chains.

Proof. We consider the decomposition

$$T = G + S, \quad G = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad S = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.$$

G is skew-adjoint with compact resolvent, $\sigma(G) = \{ir_k \mid k \in \Lambda\}$, and almost all of its eigenvalues have multiplicity 2. S is bounded with $\|S\| = b$. By Theorem 4.5 T has a compact resolvent. If z is a point outside the discs D_k with radius b

around the ir_k , then $\text{dist}(z, \sigma(G)) > b$ and $\|(G - z)^{-1}\| < b^{-1}$; thus $z \in \varrho(T)$ by Lemma 4.1.

Now we use the indefinite inner products $(J_j \cdot | \cdot)$ on $H \times H$ given by the fundamental symmetries

$$J_1 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

where $(\cdot | \cdot)$ is the standard scalar product on $H \times H$. We refer to [3] for a treatment of indefinite inner product spaces and operators therein. It is easy to verify that T is J_1 -skew-adjoint (i.e., $J_1 T$ is skew-adjoint), which implies that $\sigma(T)$ is symmetric with respect to $i\mathbb{R}$. On the other hand, for an eigenvalue λ of T with eigenvector w an easy calculation yields

$$\gamma \|w\|^2 \leq \text{Re}(J_2 T w | w) \leq |\text{Re } \lambda| |(J_2 w | w)| \leq |\text{Re } \lambda| \|w\|^2;$$

hence $|\text{Re } \lambda| \geq \gamma$, which shows the asserted shape of the spectrum.

Finally we apply Theorem 6.2 with $p = \beta_1 = \beta_2 = 0$, $\theta_1 = \pi/2$, $\theta_2 = 3\pi/2$. It shows that $N(D_k, T) = 2$ for almost all discs D_k . Consequently, almost all D_k contain only one skew-conjugate pair of simple eigenvalues $\lambda, -\bar{\lambda}$ with $|\text{Re } \lambda| \geq \gamma$. The second part of the theorem thus implies that the root subspaces of T form a Riesz basis. Since almost all root subspaces have dimension one, this is equivalent to the existence of a Riesz basis of eigenvectors and finitely many Jordan chains.

□

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